# Homotopy-commutativity in spinor groups

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## 1 Introduction

For two subsets $S$ and $S'$ of a topological group $G$ which contain the unit of $G$ as its base points, we say $S$ and $S'$ homotopy-commute in $G$, when the commutator map $c$ from $S \wedge S'$ to $G$ which maps $(x, y) \in S \wedge S'$ to $xyx^{-1}y^{-1} \in G$ is null homotopic.

In [3], the first author showed the next theorem:

**Theorem 1.1.** *Let $n$, $m$ be positive integers, and let $n + m \neq 4$ or $8$. If $n$ or $m$ is even or if $\binom{n+m-2}{n-1} \equiv 0$ mod 2 then $SO(n)$ and $SO(m)$ do not homotopy-commute in $SO(n + m - 1)$.*

In this paper, we describe the homotopy-commutativity of $Spin(n)$ and $Spin(m)$ in $Spin(n + m - 1)$.

**Definition 1.2.** *If $SO(n)$ and $SO(m)$ homotopy-commute in $SO(n + m - 1)$, we say $(n, m)$ is SO-irregular, and if not we say $(n, m)$ is SO-regular. Also, If $Spin(n)$ and $Spin(m)$ homotopy-commute in $Spin(n + m - 1)$, we say $(n, m)$ is Spin-irregular, and if not we say $(n, m)$ is Spin-regular.*

Main theorems are the followings:

**Theorem 1.3.** *Assume neither $n-1$ nor $m-1$ is a power of 2 and $n+m \neq 4$ or 8. If $n$ or $m$ is even or if $\binom{n+m-1}{n-1} \equiv 0$ mod 2 then $(n,m)$ is Spin-regular.*

For the case $n-1$ is a power of 2, we give some results as following:

**Theorem 1.4.** *Set $n=3$ and $m \equiv 1$ mod 4 then $(3,m)$ is Spin-irregular.*

**Remark 1.5.** Theorem 1.1 implies that if $m \not\equiv 1$ mod 4, $(3,m)$ is SO-regular.

**Remark 1.6.** In fact, since $Spin(5) \cong Sp(2)$ and $\pi_6(Sp(2)) \cong \pi_6(\boldsymbol{Sp}) \cong \widetilde{KSp}^{-7}(\mathrm{pt}) \cong 0$ where $\boldsymbol{Sp}$ is $\lim_{n\to\infty} Sp(n)$, the commutator map $c: Spin(3) \wedge Spin(3) \to Spin(5)$ is null homotopic and $(3,3)$ is Spin-irregular. On the other hand, Theorem 1.1 implies $(3,3)$ is SO-regular. Therefore SO-regularity and Spin-regularity is not the same.

This paper is organized as follows: In §2 we give a sufficient condition for $(n,m)$ to be Spin-regular, which is an existence of a map with an adequate property and show that ,when $n+m$ is odd, $(n,m)$ is Spin-regular. In §3 we introduce the maps $\phi_{i,j}$ : $\Omega^i \boldsymbol{BO} \wedge \Omega^j \boldsymbol{BO} \to \Omega^{i+j}\boldsymbol{BO}$ to investigate $\widetilde{KO}^{-*}(Spin(n) \wedge Spin(m))$ and in §4 investigate its induced cohomology maps and prove Theorem 1.3 for the case both $n$ and $m$ are odd. In §5 we look into the case $n$ and $m$ are even and complete the proof of Theorem 1.3 and finally in §6 we give the proof of Theorem 1.4.

# 2 Lift of commutator map

Similarly to [3], consider the next fibrations :

$$Spin(n+m-1) \xrightarrow{i} \boldsymbol{Spin} \xrightarrow{p} \boldsymbol{Spin}/Spin(n+m-1),$$

$$SO(n+m-1) \xrightarrow{j} \boldsymbol{SO} \xrightarrow{q} \boldsymbol{SO}/SO(n+m-1),$$

where $\boldsymbol{SO}$ ( resp. $\boldsymbol{Spin}$ ) is $\lim_{n\to\infty} SO(n)$ (resp. $\lim_{n\to\infty} Spin(n)$ ).

We refer to the cohomology rings of spaces which we use in this paper, that is,

$$\begin{aligned} \mathrm{H}^*(\Omega \boldsymbol{Spin}) &= \mathbb{Z}/2\mathbb{Z}[\alpha_2, \alpha_4, \alpha_6, \cdots]/(\alpha_{4k} - {\alpha_{2k}}^2), \\ \mathrm{H}^*(Spin(k)/Spin(k-l)) &= \Delta(x_{k-l}, \cdots, x_{k-1}), \\ \mathrm{H}^*(Spin(k)) &= \Delta(x_3, x_5, x_6, x_7, x_9, \cdots) \otimes \bigwedge(z). \end{aligned}$$

In the last equality, the index $i$ of $x_i$ scans all integers neither of which is not a power of 2 and $3 \leq i \leq k-1$. Also, $\deg(\alpha_{2i}) = 2i$ and $\deg(x_i) = i$.

Further, it can be easily seen that $\mathrm{H}^*(\Omega \boldsymbol{Spin}/Spin(n+m-1)) = 0$ for $* \leq n+m-3$ and $\mathrm{H}^{n+m-2}(\Omega \boldsymbol{Spin}/Spin(n+m-1)) = \mathbb{Z}/2\mathbb{Z}$ whose generator is written as $\alpha_{n+m-2}$. When $n+m$ is even, $\Omega p^*(\alpha_{n+m-2}) = \alpha_{n+m-2} \in \mathrm{H}^*(\Omega \boldsymbol{Spin})$.

From above fibrations, we can deduce the following fibration sequences.

$$\cdots \to \Omega \boldsymbol{Spin} \xrightarrow{\Omega p} \Omega(\boldsymbol{Spin}/Spin(n+m-1)) \xrightarrow{\delta_{Spin}} Spin(n+m-1) \xrightarrow{i} \boldsymbol{Spin} \xrightarrow{p} \boldsymbol{Spin}/Spin(n+m-1),$$

$$\cdots \to \Omega \boldsymbol{SO} \xrightarrow{\Omega q} \Omega(\boldsymbol{SO}/SO(n+m-1)) \xrightarrow{\delta_{SO}} SO(n+m-1) \xrightarrow{j} \boldsymbol{SO} \xrightarrow{q} \boldsymbol{SO}/SO(n+m-1).$$

Let $c_{SO}$ (resp. $c_{Spin}$) be the commutator map from $SO(n) \wedge SO(m)$ to $SO(n+m-1)$ (resp. from $Spin(n) \wedge Spin(m)$ to $Spin(n+m-1)$). Obviously we can see that $i \circ c_{Spin}$ and $j \circ c_{SO}$ are null homotopic. Thus there exists a lift of $c_{SO}$ from $SO(n) \wedge SO(m)$ to $\Omega \boldsymbol{SO}/SO(n+m-1)$ and a lift of $c_{Spin}$ from $Spin(n) \wedge Spin(m)$ to $\Omega \boldsymbol{Spin}/Spin(n+m-1)$.

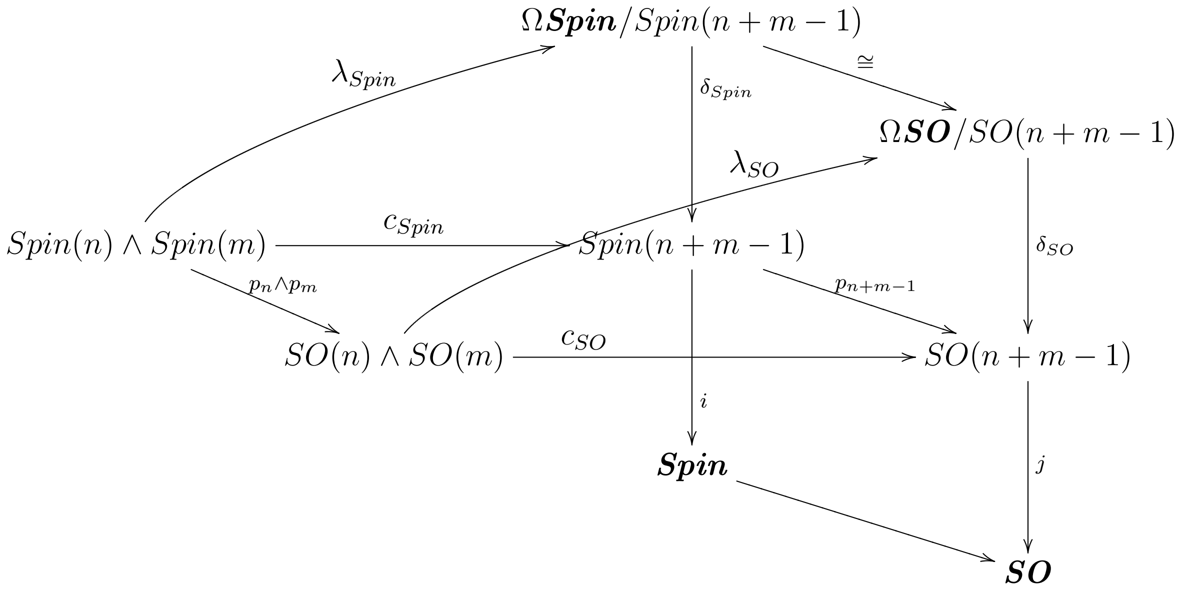
In [4], a lift of $c_{SO}$ written as $\lambda_{SO}$ was constructed and in [3], it is obtained that

$$\lambda_{SO}{}^*(\alpha_{n+m-2}) = x_{n-1} \otimes x_{m-1}. \tag{1}$$

Here set $\lambda_{Spin} = \lambda_{SO} \circ (p_n \wedge p_m)$.

**Lemma 2.1.** *$\lambda_{Spin}$ is a lift of $c_{Spin}$, that is, $\delta_{Spin} \circ \lambda_{Spin} \simeq c_{Spin}$.*

*Proof.* See the diagram below.



Since $\delta_{SO} \circ \lambda_{SO} \simeq c_{SO}$ and $\delta_{SO} \simeq p_{n+m-1} \circ \delta_{Spin}$, it occurs that

$$\begin{aligned} p_{n+m-1} \circ \delta_{Spin} \circ \lambda_{Spin} &= \delta_{SO} \circ \lambda_{SO} \circ (p_n \wedge p_m) \\ &\simeq c_{SO} \circ (p_n \wedge p_m) \\ &= p_{n+m-1} \circ c_{Spin} \end{aligned} \tag{2}$$

Now consider the fibration $\mathbb{Z}/2\mathbb{Z} \to Spin(n+m-1) \to SO(n+m-1)$. Then for any CW complex $X$ we have the exact sequence of base pointed homotopy sets:

$$[X, \mathbb{Z}/2\mathbb{Z}]_* \longrightarrow [X, Spin(n+m-1)]_* \stackrel{p_{n+m-1*}}{\longrightarrow} [X, SO(n+m-1)]_*.$$

Thus $p_{n+m-1*}$ is injective and from 2 we can see

$$\delta_{Spin} \circ \lambda_{Spin} \simeq c_{Spin}.$$

Q.E.D.

In the rest of paper, $c$, $\lambda$, $\delta$ stands for $c_{Spin}$, $\lambda_{Spin}$, $\delta_{Spin}$ respectively.

**Proposition 2.2.** *Assume neither $n-1$ nor $m-1$ is a power of 2.*

1. *If $n+m$ is odd, $c$ is not null homotopic.*
2. *Let $n+m$ is even. If for any continuous map $x$ from $Spin(n) \wedge Spin(m)$ to $\Omega\boldsymbol{Spin}$, $x^*(\alpha_{n+m-2}) \neq x_{n-1} \otimes x_{m-1}$ in cohomology, then $c$ is not null homotopic.*

*Proof.*

If $c$ is null homotopic, that is, $\delta \circ \lambda \simeq *$, then there exists a map $x : Spin(n) \wedge Spin(m) \to \Omega\boldsymbol{Spin}$ such that $\Omega p \circ x \simeq \lambda$.

From (1) we can see

$$\begin{aligned} x^*(\alpha_{n+m-2}) &= x^* \circ \Omega p^*(\alpha_{n+m-2}) \\ &= \lambda^*(\alpha_{n+m-2}) \\ &= (p_n \wedge p_m)^* \circ {\lambda_{SO}}^*(\alpha_{n+m-2}) \\ &= (p_n \wedge p_m)^*(x_{n-1} \otimes x_{m-1}) \\ &= x_{n-1} \otimes x_{m-1}, \end{aligned} \tag{3}$$

since neither $n-1$ nor $m-1$ is a power of 2. Thus the statement for the case $n+m$ is even is proved.

When $n+m$ is odd, it occurs that

$$\begin{aligned}\lambda^*(\alpha_{n+m-2}) &= x^* \circ \Omega p^*(\alpha_{n+m-2}) \\ &= x^*(0),\end{aligned}$$

since $\mathrm{H}^*(\Omega \boldsymbol{Spin})$ is concentrated in even degrees. This contradicts to (3) and $c$ is not null homotopic.

Q.E.D.

# 3 $\widetilde{KO}^{-*}(Spin(n) \wedge Spin(m))$

In this section we assume that both $n$ and $m$ are odd.

From Proposition 2.2 we should look into the homotopy set $[Spin(n) \wedge Spin(m), \Omega \boldsymbol{Spin}]$. By use of KO-theory we can say that,

$$[Spin(n) \wedge Spin(m), \Omega \boldsymbol{Spin}] \cong [Spin(n) \wedge Spin(m), \Omega_0 \boldsymbol{SO}] \cong \widetilde{KO}^{-2}(Spin(n) \wedge Spin(m)),$$

since $\Omega^2 \boldsymbol{BO} \cong \Omega \boldsymbol{SO}$.

Further more, the complex representation ring of $Spin(2k+1)$ is generated by real representations or symplectic representations. (See Proposition 6.19 in P.290 of [8].) Thus Theorem 5.12. in [11] implies that, when $n$ is odd, $KO^{-*}(Spin(n))$ is $KO^{-*}(pt)$ free. Therefore we have an decomposition of

$$\widetilde{KO}^{-*}(Spin(n) \wedge Spin(m)) \cong \widetilde{KO}^{-*}(Spin(n)) \otimes_{\widetilde{KO}^{-*}(pt)} \widetilde{KO}^{-*}(Spin(m)).$$

From now on, we identify $\widetilde{KO}^{-i}(X)$ with $[X, \Omega^i \boldsymbol{BO}]$.

**Theorem 3.1.** *There is a map $\phi_{i,j} : \Omega^i \boldsymbol{BO} \wedge \Omega^j \boldsymbol{BO} \to \Omega^{i+j} \boldsymbol{BO}$ such that for any CW-complexes $X$, $X'$ and $\alpha \in \widetilde{KO}^{-i}(X)$ and $\beta \in \widetilde{KO}^{-j}(X')$,*

$$\alpha \hat{\otimes} \beta = \phi_{i,j} \circ (\alpha \wedge \beta) \qquad \text{in } \widetilde{KO}^{-(i+j)}(X \wedge X').$$

*Proof.* First we construct $\phi_{i,j}$. Let $\xi_n$ be the universal vector bundle over $BO(n)$ and put $\eta_n = \xi_n - n$, $\eta_\infty = \lim_{n\to\infty} \eta_n$. And set $\phi_{0,0} : \boldsymbol{BO} \wedge \boldsymbol{BO} \to \boldsymbol{BO}$ as the classifying map of $\eta_\infty \hat{\otimes} \eta_\infty$. Let $\kappa_i : \Sigma^i \Omega^i \boldsymbol{BO} \to \boldsymbol{BO}$ be the map which satisfies

$$\mathrm{Ad}^i \kappa_i \simeq \mathrm{Id}_{\Omega^i \boldsymbol{BO}}.$$

Consider the composition of $\kappa_i \wedge \kappa_j$ and $\phi_{0,0}$:

$$\Sigma^i\Omega^i \boldsymbol{BO} \wedge \Sigma^j\Omega^j \boldsymbol{BO} \overset{\kappa_i \wedge \kappa_j}{\longrightarrow} \boldsymbol{BO} \wedge \boldsymbol{BO} \overset{\phi_{0,0}}{\longrightarrow} \boldsymbol{BO}.$$

We define $\phi_{i,j}$ as

$$\phi_{i,j} = \mathrm{Ad}^{i+j}(\phi_{0,0} \circ (\kappa_i \wedge \kappa_j)) : \Omega^i \boldsymbol{BO} \wedge \Omega^j \boldsymbol{BO} \longrightarrow \Omega^{i+j} \boldsymbol{BO}.$$

Now, take $\alpha \in [X, \Omega^i \boldsymbol{BO}]$ and $\beta \in [X', \Omega^j \boldsymbol{BO}]$ and see the composition of $\alpha \wedge \beta$ and $\phi_{i,j}$:

$$\phi_{i,j} \circ (\alpha \wedge \beta) : X \wedge X' \to \Omega^i \boldsymbol{BO} \wedge \Omega^j \boldsymbol{BO} \to \Omega^{i+j} \boldsymbol{BO}.$$

Taking $\mathrm{Ad}^{-(i+j)}$ of the above composition, we obtain

$$\mathrm{Ad}^{-(i+j)}(\phi_{i,j} \circ (\alpha \wedge \beta)) = (\mathrm{Ad}^{-(i+j)}\phi_{i,j}) \circ (\Sigma^i \alpha \wedge \Sigma^j \beta)$$

$$: \Sigma^{i+j}(X \wedge X') \to \Sigma^{i+j}(\Omega^i \boldsymbol{BO} \wedge \Omega^j \boldsymbol{BO}) \to \boldsymbol{BO}.$$

From definition of $\phi_{i,j}$, $\mathrm{Ad}^{-(i+j)}(\phi_{i,j} \circ (\alpha \wedge \beta))$ is the composition of following maps:

$$\Sigma^{i+j}(X \wedge X') \overset{\Sigma^i \alpha \wedge \Sigma^j \beta}{\longrightarrow} \Sigma^{i+j}(\Omega^i \boldsymbol{BO} \wedge \Omega^j \boldsymbol{BO}) \overset{\kappa_i \wedge \kappa_j}{\longrightarrow} \boldsymbol{BO} \wedge \boldsymbol{BO} \overset{\phi_{0,0}}{\longrightarrow} \boldsymbol{BO}. \tag{4}$$

**Lemma 3.2.** *For any continuous map $f : \Sigma^i X \to \boldsymbol{BO}$,*

$$f \simeq \kappa_i(\Sigma^i Ad^i f).$$

*Proof.* Consider the composition of $\mathrm{Ad}^i f$ and identity map of $\Omega^i \boldsymbol{BO}$.

$$X \overset{\mathrm{Ad}^i f}{\longrightarrow} \Omega^i \boldsymbol{BO} \overset{\mathrm{Id}_{\Omega^i \boldsymbol{BO}}}{\longrightarrow} \Omega^i \boldsymbol{BO}.$$

Taking $\mathrm{Ad}^{-i}$ of the above composition, we have

$$f = \mathrm{Ad}^{-i}(\mathrm{Id}_{\Omega^i \boldsymbol{BO}} \circ \mathrm{Ad}^i f) = \kappa_i \circ \Sigma^i \mathrm{Ad}^i f$$

$$: \Sigma^i X \overset{\Sigma^i \mathrm{Ad}^i f}{\longrightarrow} \Sigma^i \Omega^i \boldsymbol{BO} \overset{\kappa_i}{\longrightarrow} \boldsymbol{BO}.$$

Q.E.D.

By (4) and the above lemma, it follows that

$$\begin{aligned}\mathrm{Ad}^{-(i+j)}(\phi_{i,j}\circ(\alpha\wedge\beta)) &\simeq \phi_{0,0}\circ(\kappa_i\wedge\kappa_j)\circ(\Sigma^i\alpha\wedge\Sigma^j\beta)\\ &\simeq \phi_{0,0}\circ(\kappa_i\circ\Sigma^i\alpha)\wedge(\kappa_j\circ\Sigma^j\beta)\\ &\simeq \phi_{0,0}\circ(\mathrm{Ad}^{-i}\alpha\wedge\mathrm{Ad}^{-j}\beta).\end{aligned}$$

Since $f\in[X,\Omega^i\boldsymbol{BO}]$ corresponds to $(\mathrm{Ad}^{-i}f)^*(\eta_\infty)\in\widetilde{KO}^{-i}(X)$, the above equation says that $\phi_{i,j}\circ(\alpha\wedge\beta)$ corresponds to

$$(\mathrm{Ad}^{-i}\alpha\wedge\mathrm{Ad}^{-j}\beta)^*{\phi_{0,0}}^*(\eta_\infty)=\mathrm{Ad}^{-i}\alpha^*(\eta_\infty)\hat{\otimes}\mathrm{Ad}^{-j}\beta^*(\eta_\infty).$$

Therefore we obtain that

$$\alpha\hat{\otimes}\beta=\phi_{i,j}\circ(\alpha\wedge\beta)\qquad\text{in }\widetilde{KO}^{-(i+j)}(X\wedge X').$$

Q.E.D.

From the above theorem, we can deduce the next theorem.

**Theorem 3.3.** *Assume both $n$ and $m$ are odd. If, for all $(i,j)\in\mathbb{Z}/8\mathbb{Z}^2$ which satisfy $i+j=2$, ${\phi_{i,j}}^*(\alpha_{n+m-2})=\sum b_s\otimes c_t$ where $|b_s|=s$ and $|c_t|=t$ and $b_{n-1}\otimes c_{m-1}=0$ then $c:Spin(n)\wedge Spin(m)\to Spin(n+m-1)$ is not null homotopic.*

*Proof.* For any $\eta\in\widetilde{KO}^{-2}(Spin(n)\wedge Spin(m))$, there exist $\alpha_a\in\widetilde{KO}^{-i_a}(Spin(n))$ and $\beta_a\in\widetilde{KO}^{-j_a}(Spin(m))$ such that $\eta=\sum\alpha_a\hat{\otimes}\beta_a$ and $i_a+j_a=2$. Since $\alpha_{n+m-2}$ is primitive,

$$\eta^*(\alpha_{n+m-2})=(\sum\alpha_a\hat{\otimes}\beta_a)^*(\alpha_{n+m-2})=\sum(\alpha_a\hat{\otimes}\beta_a)^*(\alpha_{n+m-2})$$

and by Theorem 3.1,

$$(\alpha\hat{\otimes}\beta)^*(\alpha_{n+m-2})=(\alpha\wedge\beta)^*\circ{\phi_{i,j}}^*(\alpha_{n+m-2}).$$

If the hypothesis is satisfied, $\eta^*(\alpha_{n+m-2})$ can not be $x_{n-1}\otimes x_{m-1}$. Therefore from Proposition 2.2, $c$ is not null homotopic.

Q.E.D.

# 4 the case $n$ and $m$ are odd

In this section we investigate the induced cohomology map of $\phi_{i,j}$ for $(i, j) \in (\mathbb{Z}/8\mathbb{Z})^2$, such that, $i + j = 2$.

We start from the next lemma.

**Lemma 4.1.** *Assume $a \in \mathrm{H}^u(\Omega^{i+j}\boldsymbol{BO})$ is primitive and ${\phi_{i,j}}^*(a) = \sum_{s+t=u} b_s \otimes c_t$ where $|b_s| = s$ and $|c_t| = t$. Then $b_s$ and $c_t$ are primitive.*

*Proof.* Since for any $\alpha, \beta, \gamma \in \widetilde{KO}(X)$,

$$(p_1^*(\alpha) \oplus p_2^*(\beta)) \otimes p_3^*(\gamma) = (p_1^*(\alpha) \otimes p_3^*(\gamma)) \oplus (p_2^*(\beta) \otimes p_3^*(\gamma))$$

where $p_i : X \times X \times X \to X$ is the projection to $i$-th component, the next diagram commutes.

$$\begin{array}{ccc}
\Omega^i\boldsymbol{BO} \times \Omega^i\boldsymbol{BO} \times \Omega^i\boldsymbol{BO} & \xrightarrow{\Omega^i\mu\times 1} & \Omega^i\boldsymbol{BO} \times \Omega^j\boldsymbol{BO} \\
\downarrow{\scriptstyle (1\times T\times 1)\circ(1\times 1\times\Delta)} & & \\
\Omega^i\boldsymbol{BO} \times \Omega^j\boldsymbol{BO} \times \Omega^i\boldsymbol{BO} \times \Omega^j\boldsymbol{BO} & & \downarrow{\scriptstyle \hat{\phi}_{i,j}} \\
\downarrow{\scriptstyle \hat{\phi}_{i,j}\times\hat{\phi}_{i,j}} & & \\
\Omega^{i+j}\boldsymbol{BO} \times \Omega^{i+j}\boldsymbol{BO} & \xrightarrow{\Omega^{i+j}\mu} & \boldsymbol{BO}
\end{array}$$

Here $T$ is the transposition map, $\Delta$ is the diagonal map and $\mu : \boldsymbol{BO} \times \boldsymbol{BO} \to \boldsymbol{BO}$ is the classifying map of $\eta_\infty \times \eta_\infty$ over $\boldsymbol{BO} \times \boldsymbol{BO}$. Further, $\hat{\phi}_{i,j}$ is the next composition:

$$\Omega^i\boldsymbol{BO} \times \Omega^j\boldsymbol{BO} \to \Omega^i\boldsymbol{BO} \wedge \Omega^j\boldsymbol{BO} \to \Omega^{i+j}\boldsymbol{BO}.$$

Let $a \in \mathrm{H}^u(\Omega^{i+j}\boldsymbol{BO})$ be a primitive element. Then we have

$$\begin{aligned}
& (1 \otimes \Delta^*) \circ (1 \otimes T^* \otimes 1) \circ ({\hat{\phi}_{i,j}}^* \otimes {\hat{\phi}_{i,j}}^*) \circ \mu^*(a) \\
= \ & (1 \otimes \Delta^*) \circ (1 \otimes T^* \otimes 1) \circ ({\hat{\phi}_{i,j}}^* \otimes {\hat{\phi}_{i,j}}^*)(a \otimes 1 + 1 \otimes a) \\
= \ & (1 \otimes \Delta^*) \circ (1 \otimes T^* \otimes 1)(\sum b_s \otimes c_t \otimes 1 \otimes 1 + \sum 1 \otimes 1 \otimes b_s \otimes c_t) \\
= \ & (1 \otimes \Delta^*)(\sum b_s \otimes 1 \otimes c_t \otimes 1 + \sum 1 \otimes b_s \otimes 1 \otimes c_t) \\
= \ & (\sum b_s \otimes 1 \otimes c_t + \sum 1 \otimes b_s \otimes c_t) \\
= \ & (\sum (b_s \otimes 1 + 1 \otimes b_s) \otimes c_t).
\end{aligned}$$

Also

$$(\mu^* \otimes 1) \circ \hat{\phi}_{i,j}{}^*(a) = (\mu^* \otimes 1)(\sum b_s \otimes c_t)$$
$$= \sum \mu^*(b_s) \otimes c_t.$$

The above diagram says that these are the same. Therefore it occurs that $\mu^*(b_s) = b_s \otimes 1 + 1 \otimes b_s$, that is, $b_s$ is primitive. Similarly we can prove that $c_t$ is primitive.

Q.E.D.

**Theorem 4.2.** *Let $i + j = 2$ and $n$ and $m$ be odd. Assume $\phi_{i,j}(\alpha_{n+m-2}) = \sum b_s \otimes c_t$ where $|b_s| = s$ and $|c_t| = t$. If $\binom{n+m-2}{n-1} \equiv 0$ mod 2, then $b_{n-1} \otimes c_{m-1} = 0$.*

*Proof.* From assumption, $(i, j)$ is $(1, 1)$, $(2, 0)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$ or $(0, 2)$. From the symmetricity, we shall look in to the cases $(i, j) = (1, 1)$, $(2, 0)$, $(3, 7)$, $(4, 6)$ and $(5, 5)$.

For $\phi_{3,7}$, $\phi_{5,5}$, the proof is easy. From the assumption, $n-1$ and $m-1$ are even and by Lemma 4.1, $b_{n-1}$ and $c_{m-1}$ are primitive. On the other hand, it is known that all of the non-zero primitive elements of $\Omega^3\boldsymbol{BO}$, $\Omega^5\boldsymbol{BO}$ are in odd degrees.[7] Thus $b_{n-1} \otimes c_{m-1} = 0$.

To start the proof for $\phi_{2,0}$, we investigate $\phi_{0,0}{}^*$.

Let $N = 2^r$, $r \in \mathbf{N}$ and $\eta \in \widetilde{KO}(\boldsymbol{BO}(N) \wedge \boldsymbol{BO}(N))$ be the class of

$$\eta = (\xi_N - N)\hat{\otimes}(\xi_N - N).$$

We calculate the total Stiefel-Whitney class of $\eta$ in $\mathrm{H}^*(\mathbf{B}(\mathbb{Z}/2\mathbb{Z})^N \wedge \mathbf{B}(\mathbb{Z}/2\mathbb{Z})^N) \supset \mathrm{H}^*(\boldsymbol{BO} \wedge \boldsymbol{BO})$. Let $t_1, \cdots, t_N$ and $t'_1, \cdots, t'_N$ be the generator of $\mathrm{H}^*(\mathbf{B}(\mathbb{Z}/2\mathbb{Z})^N \wedge \mathbf{B}(\mathbb{Z}/2\mathbb{Z})^N)$ where $t_i$ corresponds to the first component and $t'_i$ corresponds to the second. Then $w_k = \sigma_k(t_1, \cdots, t_N)$ and $w'_k = \sigma_k(t'_1, \cdots, t'_N)$ $(1 \leq k \leq N)$ are the generators of $\mathrm{H}^*(\boldsymbol{BO} \wedge \boldsymbol{BO})$ where $\sigma_k$ is $k$-th fundamental symmetric polynomial. (We put $w_0 = 1$.) Also we set $S'_l = \sum_{i=1}^{N} {t'_i}^l$.

**Lemma 4.3.** *The total Stiefel-Whitney class of $\eta$ satisfies*

$$w(\eta) = 1 + \sum_{k=0}^{N-1} \sum_{l=0}^{k} \binom{N-k}{l} w_{N-k} \otimes S'_l \quad modulo \quad (w_1 \otimes 1, w_2 \otimes 1, \cdots, w_N \otimes 1)^2$$

*in* $\mathrm{H}^*(\boldsymbol{BO}(N) \wedge \boldsymbol{BO}(N))$ *for* $* < N$.

*Proof.* Since

$$\eta = \xi_N \hat{\otimes} \xi_N - \xi_N \hat{\otimes} N - N \hat{\otimes} \xi_N + N \hat{\otimes} N,$$

we can see that

$$w(\eta) = \prod_{1 \le i \le N,\, 1 \le j \le N} (1 + t_i + t'_j) \prod_{1 \le i \le N} (1 + t_i)^{-N} \prod_{1 \le j \le N} (1 + t'_j)^{-N}.$$

Here in the part of degrees less than $N$, $(1+t_i)^{-N} = (1+{t_i}^N)^{-1} = 1$ and also $(1+t'_j)^{-N} = 1$. Therefore modulo $\bigoplus_{i \ge N} \mathrm{H}^i(\mathbf{B}(\mathbb{Z}/2\mathbb{Z})^N \times \mathbf{B}(\mathbb{Z}/2\mathbb{Z})^N)$, we obtain that

$$\begin{aligned} w(\eta) &= \prod_{1 \le i \le N,\, 1 \le j \le N} (t_i + 1 + t'_j) \\ &= \prod_{j=1}^{N} \Big(\sum_{k=0}^{N} w_k (1 + t'_j)^{N-k}\Big) \\ &= \prod_{j=1}^{N} \Big(1 + \sum_{k=1}^{N} \sum_{l=0}^{N-k} \tbinom{N-k}{l} w_k {t'_j}^l\Big). \end{aligned}$$

We proceed the calculation modulo $(w_1 \otimes 1, w_2 \otimes 1, \cdots, w_N \otimes 1)^2$ and obtain

$$\begin{aligned} w(\eta) &\equiv 1 + \sum_{k=1}^{N} \sum_{l=1}^{N-k} \tbinom{N-k}{l} w_k S'_l \\ &\equiv 1 + \sum_{1 \le k, 1 \le l,\, k+l \le N} \tbinom{N-k}{l} w_k S'_l. \end{aligned}$$

Q.E.D.

**Lemma 4.4.** *Let $k$, $l$, $r \in \mathbf{N}$. If $2^r > k + l$, then $\binom{2^r - k}{l} \equiv \binom{k+l-1}{l}$ mod 2.*

*Proof.* We set the binary expansion of $k-1$, $l$ as

$$k - 1 = \sum_{0 \le i \le r-1} \epsilon_i 2^i, \quad l = \sum_{0 \le i \le r-1} \delta_i 2^i.$$

Then we have

$$\binom{2^r - k}{l} = \binom{(2^r - 1) - (k-1)}{l} \equiv \prod_{0 \le i \le r-1} \binom{1 - \epsilon_i}{\delta_i}.$$

Therefore $\binom{2^r - k}{l} \equiv 0$ if and only if, for some $i$, $\binom{1-\epsilon_i}{\delta_i} \equiv 0$, i.e., $\epsilon_i = \delta_i = 1$.

Assume, for some $i$ $(0 \leq i \leq r-1)$, $\epsilon_i = \delta_i = 1$. Then let $i_0$ be the smallest such a number. Then $i_0$-th coefficient of the binary expansion of $k+l-1$ is 0, while $\delta_{i_0} = 1$. Thus we have $\binom{k+l-1}{l} \equiv 0$.

Vice versa if, for any $i$ $(0 \leq i \leq r-1)$, not both $\epsilon_i$ and $\delta_i$ are 1, then

$$\binom{k+l-1}{l} \equiv \prod_{0 \leq i \leq r-1} \binom{\epsilon_i + \delta_i}{\delta_i} \not\equiv 0.$$

Therefore $\binom{2^r - k}{l} \equiv \binom{k+l-1}{l} \bmod 2$.

Q.E.D.

Since $\phi_{0,0}$ is the classifying map of $\eta_\infty \hat{\otimes} \eta_\infty$, Lemma 4.3 implies that

$$\begin{aligned} {\phi_{0,0}}^*(w_i) &= \sum_{k+l=i} \binom{2^r-k}{l} w_k \otimes S'_l \\ &= \sum_{k+l=i} \binom{k+l-1}{l} w_k \otimes S'_l \quad \text{modulo} \quad (w_1 \otimes 1, w_2 \otimes 1, w_3 \otimes 1, \cdots)^2 \qquad (5) \end{aligned}$$

where $r$ is sufficiently large.

Therefore

$$(\kappa_2 \wedge \mathrm{Id}_{\boldsymbol{BO}})^* \circ {\phi_{0,0}}^*(w_i) = \sum_{k+l=i,\ k:\text{even}} \binom{k+l-1}{l} \Sigma^2 a_{k-2} \otimes S'_l,$$

since

$${\kappa_2}^*(w_k) = \begin{cases} \Sigma^2 a_{k-2} & k : \text{even} \\ 0 & k : \text{odd} \end{cases}$$

and ${\kappa_2}^*(\text{decomposable element}) = 0$.

From definition, $\phi_{2,0} = \mathrm{Ad}^2(\kappa_2 \wedge \mathrm{Id} \circ \phi_{0,0})$ and then we have

$${\phi_{2,0}}^*(a_{4i+2}) = \sum_{k+l=4i+2,\ k:\text{even}} \binom{k+l}{l} a_k \otimes S_l, \qquad (6)$$

here we remark that $\binom{k+l+1}{l} = \binom{k+l}{l}$ when $k$ and $l$ are even. From (6), and since $a_{4k} = {a_{2k}}^2$, it occurs that

$${\phi_{2,0}}^*(a_{2^p(4i+2)}) = \sum_{k+l=4i+2,\ k:\text{even}} \binom{k+l}{l} {a_k}^{2^p} \otimes {S_l}^{2^p},$$

Thus the coefficient of $b_{n-1} \otimes c_{m-1}$ in ${\phi_{2,0}}^*(a_{n+m-2})$ is 0 when $\binom{n+m-2}{n-1} = 0$ and the statement is true for $\phi_{2,0}$.

Second case is $\phi_{1,1}$. Consider the composition of following maps.

$$\Sigma\Omega \boldsymbol{BO} \wedge \Sigma\Omega \boldsymbol{BO} \stackrel{\kappa_1 \wedge \kappa_1}{\longrightarrow} \boldsymbol{BO} \wedge \boldsymbol{BO} \stackrel{\phi_{0,0}}{\longrightarrow} \boldsymbol{BO}.$$

From (5) and since ${\kappa_1}^*(\text{decomposable element}) = 0$ and

$$\begin{aligned} {\kappa_1}^*(w_k) &= \Sigma x_{k-1} \\ {\kappa_1}^*(S_l) &= \begin{cases} \Sigma x_{l-1} & k : \text{odd} \\ 0 & k : \text{even}, \end{cases} \end{aligned}$$

the induced cohomology map of this composition can be obtained as

$$(\kappa_1 \wedge \kappa_1)^* \circ {\phi_{0,0}}^*(w_i) = (\kappa_1 \wedge \kappa_1)^*(\sum_{k+l=i} \tbinom{k+l-1}{l} S_l \otimes w'_k) \tag{7}$$

$$= \sum_{k+l=i,\ l:\ \text{odd}} \tbinom{k+l-1}{l} \Sigma x_{l-1} \otimes \Sigma x_{k-1}. \tag{8}$$

Here we remark that $\binom{k+l-1}{l} = 0$ when $l$ is odd and $k$ is even. Thus it occurs that

$$(\kappa_1 \wedge \kappa_1)^* \circ {\phi_{0,0}}^*(w_i) = \sum_{k+l=i,\ l:\ \text{odd},\ k:\ \text{odd}} \tbinom{k+l-1}{l} \Sigma x_{l-1} \otimes \Sigma x_{k-1}. \tag{9}$$

Similarly as the case of $\phi_{2,0}$, $\phi_{1,1} = \text{Ad}^2(\kappa_1 \wedge \kappa_1 \circ \phi_{0,0})$ and from (9) we have

$$\begin{aligned} {\phi_{1,1}}^*(\alpha_{4i+2}) &= \sum_{k+l=4(i+1),\ l:\ \text{odd},\ k:\ \text{odd}} \tbinom{k+l-1}{l} x_{l-1} \otimes x_{k-1} \\ &= \sum_{k+l=4i+2,\ l:\ \text{even},\ k:\ \text{even}} \tbinom{k+l}{l} x_l \otimes x_k. \end{aligned} \tag{10}$$

And also

$$\phi_{1,1}{}^*(\alpha_{2^p(4i+2)}) = \sum_{k+l=4i+2,\ l:\ \text{even},\ k:\ \text{even}} \tbinom{k+l}{l} {x_l}^{2^p} \otimes {x_k}^{2^p}. \tag{11}$$

Thus the coefficient of $b_{n-1} \otimes c_{m-1}$ in ${\phi_{1,1}}^*(a_{n+m-2})$ is also 0 when $\binom{n+m-2}{n-1} = 0$ and the statement is true for $\phi_{1,1}$.

The final case is $\phi_{4,6}$. Let $\xi_n^{\mathbf{R}}$, $\xi_n^{\mathbf{C}}$ and $\xi_n^{\mathbf{H}}$ be the universal bundle over $BO(n)$, $BU(n)$ and $BSp(n)$ respectively and put

$$\eta_n^{\mathbf{R}} = \xi_n^{\mathbf{R}} - n,\ \eta_n^{\mathbf{C}} = \xi_n^{\mathbf{C}} - n,\ \eta_n^{\mathbf{H}} = \xi_n^{\mathbf{H}} - n.$$

and

$$\eta_\infty^{\mathbf{R}} = \lim_{n\to\infty} \xi_n^{\mathbf{R}} - n,\ \eta_\infty^{\mathbf{C}} = \lim_{n\to\infty} \xi_n^{\mathbf{C}} - n,\ \eta_\infty^{\mathbf{H}} = \lim_{n\to\infty} \xi_n^{\mathbf{H}} - n.$$

Also set $c$ be the classifying map of $(\eta_\infty^{\mathbf{R}})_{\mathbf{C}}$, complexification of $\eta_\infty^{\mathbf{R}}$, $c'$ be the classifying map of $\eta_\infty^{\mathbf{H}}$ as a complex vector bundle and $\psi$ be the classifying map of $\eta_\infty^{\mathbf{C}}\hat{\otimes}\eta_\infty^{\mathbf{C}}$ over $\boldsymbol{BU}\wedge\boldsymbol{BU}$.

We start from the next lemma.

**Lemma 4.5.** *The next diagram commutes.*

$$\begin{array}{ccc} \boldsymbol{BSp}\wedge\boldsymbol{BSp} & \xrightarrow{c'\wedge c'} & \boldsymbol{BU}\wedge\boldsymbol{BU} \\ \downarrow{\phi_{4,4}} & & \downarrow{\psi} \\ \boldsymbol{BO} & \xrightarrow{c} & \boldsymbol{BU} \end{array}$$

*Proof.* Consider the next composition:

$$\Sigma^4\boldsymbol{BSp}\wedge\Sigma^4\boldsymbol{BSp}\overset{\kappa_4\wedge\kappa_4}{\longrightarrow}\boldsymbol{BO}\wedge\boldsymbol{BO}\overset{\phi_{0,0}}{\longrightarrow}\boldsymbol{BO}\overset{c}{\longrightarrow}\boldsymbol{BU}.$$

Here in K-theory, $c^*(\eta_\infty^{\mathbf{C}}) = (\eta_\infty^{\mathbf{R}})_{\mathbf{C}}$ and ${\phi_{0,0}}^*((\eta_\infty^{\mathbf{R}})_{\mathbf{C}}) = (\eta_\infty^{\mathbf{R}})_{\mathbf{C}}\hat{\otimes}(\eta_\infty^{\mathbf{R}})_{\mathbf{C}}$. Also it is known that ${\kappa_4}^*((\eta_\infty^{\mathbf{R}})_{\mathbf{C}}) = (\zeta_{\mathbf{H}} - \mathbb{H})\hat{\otimes}_{\mathbb{C}}\eta_\infty^{\mathbf{H}}$ where $\zeta_{\mathbf{H}}$ is the $\mathbb{H}$ canonical line bundle over $\mathbb{H}P^1$. Therefore above composition pulls back $\eta_\infty^{\mathbf{C}}$ to $(\zeta_{\mathbf{H}}-\mathbb{H})\hat{\otimes}_{\mathbb{C}}(\zeta_{\mathbf{H}}-\mathbb{H})\hat{\otimes}_{\mathbb{C}}\eta_\infty^{\mathbf{H}}\hat{\otimes}_{\mathbb{C}}\eta_\infty^{\mathbf{H}}$.

On the other hand consider the next composition:

$$\Sigma^8\boldsymbol{BSp}\wedge\boldsymbol{BSp}\overset{\Sigma^8(c'\wedge c')}{\longrightarrow}\Sigma^8\boldsymbol{BU}\wedge\boldsymbol{BU}\overset{\Sigma^8\psi}{\longrightarrow}\Sigma^8\boldsymbol{BU}\overset{\kappa_8'}{\longrightarrow}\boldsymbol{BU}.$$

Here $\kappa_8'$ is defined as follows. From Bott Periodicity, we know that $\Omega^2\boldsymbol{BU}\cong\boldsymbol{BU}\times\mathbb{Z}$. Thus there exists a map $\kappa_{2i}' : \Sigma^{2i}\boldsymbol{BU}\to\boldsymbol{BU}$ which satisfies $\mathrm{Ad}^{2i}\kappa_{2i}'$ is the inclusion map $\boldsymbol{BU}\to\Omega^{2i}\boldsymbol{BU}$. One can easily verify that

$$\kappa_2'\circ\Sigma^2\kappa_2'\circ\cdots\Sigma^{2i-2}\kappa_2'\simeq\kappa_{2i}'$$

and it is known that in K-theory ${\kappa_2'}^*(\eta_\infty^{\mathbf{C}}) = (\zeta_{\mathbf{C}}-\mathbb{C})\hat{\otimes}\eta_\infty^{\mathbf{C}}$ where $\zeta_{\mathbf{C}}$ is the canonical line bundle over $\mathbb{C}P^1$. Therefore ${\kappa_8'}^* = (\zeta_{\mathbf{C}}-\mathbb{C})^4\hat{\otimes}\eta_\infty^{\mathbf{C}}$. Now we can see that the above composition pulls back $\eta_\infty^{\mathbf{C}}$ to $(\zeta_{\mathbf{C}}-\mathbb{C})^4\hat{\otimes}\eta_\infty^{\mathbf{H}}\hat{\otimes}_{\mathbb{C}}\eta_\infty^{\mathbf{H}}$.

Since $\widetilde{K}^{-4}(\mathrm{pt}) = \mathbb{Z}$ and the second Chern class of $-(\zeta_{\mathbf{H}} - \mathbb{H})$ and $(\zeta_{\mathbf{C}} - \mathbb{C})^2$ coincide, we see that the above two compositions are homotopic each other.

Take the $\mathrm{Ad}^8$ of two compositions and we obtain

$$c \circ \phi_{4,4} \simeq \psi \circ c'$$

Q.E.D.

Refer to the diagram of Lemma 4.5. We want to calculate $\phi_{4,4}(w_i)$. As we have done in the proof of Lemma 4.3, let $N = 2^r$, $r \in \mathbf{N}$ and $\theta \in \widetilde{K}(BU(2N) \times BU(2N))$ be the class of $\theta = (\xi_{2N}^{\mathbf{C}} - 2N)\hat{\otimes}(\xi_{2N}^{\mathbf{C}} - 2N)$ where $\xi_{2N}^{\mathbf{C}}$ is the universal vector bundle over $BU(2N)$. Also let $\psi_N$ be the classifying map of $\theta$.

First, we calculate the total Chern class of $\theta$ in $\mathrm{H}^*(BT^{2N} \times BT^{2N}) \supset \mathrm{H}^*(BU(2N) \times BU(2N))$. Let $t_1, \cdots, t_{2N}, t'_1, \cdots, t'_{2N} \in \mathrm{H}^*(BT^{2N} \times BT^{2N})$ be the generators as usual. Then in the part of degree less than $4N$,

$${\psi_N}^*(1 + \sum_{i=1}^{\infty} c_i) = \prod_{1 \le i \le 2N, 1 \le j \le 2N} (1 + t_i + t'_j).$$

Now we proceed the calculations of $(c' \wedge c')^* {\psi_N}^*(1 + \sum_{i=1}^{\infty} c_i)$ in $\mathrm{H}^*(BT^N \times BT^N) \supset \mathrm{H}^*(BSp(N) \times BSp(N))$. Let $s_1, \cdots, s_N, s'_1, \cdots, s'_N \in \mathrm{H}^*(BT^N \times BT^N)$ be the generators. Then we can see

$$\begin{aligned}
(c' \wedge c')^* {\psi_N}^*(1 + \sum_{i=1}^{\infty} c_i) &= (c' \wedge c')^* (\prod_{1 \le i \le 2N, 1 \le j \le 2N} (1 + t_i + t'_j)) \\
&= \prod_{1 \le i \le N, 1 \le j \le N} (1 + s_i + s'_j)(1 + s_i - s'_j)(1 - s_i + s'_j)(1 - s_i - s'_j) \\
&= \prod_{1 \le i \le N, 1 \le j \le N} (1 + s_i + s'_j)^4 \\
&= \{ \prod_{1 \le i \le N,\ 1 \le j \le N} (1 + {s_i}^2 + {s'_j}^2) \}^2.
\end{aligned}$$

On the other hand, considering $\mathrm{H}^*(BSp(N)) \subset \mathrm{H}^*(\boldsymbol{BSp})$, in the part of degree less than $4N$,

$$\begin{aligned}(c' \wedge c')^* {\psi_N}^*(1+\sum_{i=1}^{\infty} c_i) &= {\phi_{4,4}}^* c^*(1+\sum_{i=1}^{\infty} c_i) \\ &= {\phi_{4,4}}^*(1+\sum_{i=1}^{\infty} w_i^2) \\ &= {\phi_{4,4}}^*(1+\sum_{i=1}^{\infty} w_i)^2\end{aligned}$$

Since $\mathrm{H}^*(\boldsymbol{BSp} \wedge \boldsymbol{BSp})$ is a subalgebra of a polynomial algebra, the square of any element in $\mathrm{H}^*(\boldsymbol{BSp} \wedge \boldsymbol{BSp})$ does not vanishes. Therefore

$${\phi_{4,4}}^*(1+\sum_{i=1}^{\infty} w_i) = \{\prod_{1\le i\le N, 1\le j\le N} (1+{s_i}^2+{s'_j}^2)\}^2.$$

in the part of degree less than $2N$.

We set $q'_k = \sigma_k({s'_1}^2, \cdots, {s'_N}^2)$ $(1 \le k \le N)$ which are the generators of $\mathrm{H}^*(BSp(N))$ and $Q_l = \sum_{i=1}^N {s_i}^{2l}$ which is the primitive element of $\mathrm{H}^*(BSp(N))$. Now we have in the part of degrees less than $2N$

$$\begin{aligned}{\phi_{4,4}}^*(1+\sum_{i=1}^{\infty} w_i) &= \prod_{1\le i\le N, 1\le j\le N} (1+{s_i}^2+{s'_j}^2) \\ &= \prod_{i=1}^{N}(\sum_{k=0}^{N}(1+{s_i}^2)^k q'_{N-k}) \\ &= \prod_{i=1}^{N}(1+\sum_{k=0}^{N-1}\sum_{l=0}^{k}\binom{k}{l}{s_i}^{2l}q'_{N-k})\end{aligned}$$

Now we proceed the calculations modulo $(q'_1, \cdots, q'_N)^2$.

$$\begin{aligned}{\phi_{4,4}}^*(1+\sum_{i=1}^{\infty} w_i) &\equiv 1+\sum_{k=0}^{N-1}\sum_{l=1}^{k}\binom{k}{l}Q_l q'_{N-k} \\ &\equiv 1+\sum_{k=1}^{N}\sum_{l=1}^{N-k}\binom{N-k}{l}Q_l q'_k \\ &\equiv 1+\sum_{i=1}^{N}\sum_{1\le k, 1\le l, k+l=i}\binom{N-k}{l}Q_l q'_k\end{aligned}$$

This leads us to the next lemma.

**Lemma 4.6.** *Modulo* $(1\otimes q_1, 1\otimes q_2, 1\otimes q_3,\cdots)^2$,

$$\phi_{4,4}{}^*(w_i) = \begin{cases} \sum_{1\le k,1\le l,k+l=j} \binom{k+l-1}{l} Q_l \otimes q_k & i=4j \\ 0 & i \not\equiv 0 \ mod\ 4 \end{cases}$$

*where* $\mathrm{H}^*(\boldsymbol{BSp}) = \mathbb{Z}/2\mathbb{Z}[q_1,q_2,q_3,\cdots]$ *and* $Q_l \in \mathrm{H}^*(\boldsymbol{BSp})$ *is the primitive element of degree* $4l$*.*

Let $\kappa' : \Sigma^2\Omega^6\boldsymbol{BO} \to \Omega^4\boldsymbol{BO}$ be the map which satisfies $\mathrm{Ad}^2(\kappa') = \mathrm{Id}_{\Omega^6\boldsymbol{BO}}$. Then it can be easily verified that $\mathrm{Ad}^2(\phi_{4,4}\circ \mathrm{Id}_{\Omega^4\boldsymbol{BO}}\wedge\kappa') = \phi_{4,6}$. Since

$$\kappa'^*(q_l) = \Sigma^2 b_{4l-2},$$

where $\mathrm{H}^*(\Omega^2\boldsymbol{BSp}) = \bigwedge(b_2,b_4,b_6,\cdots)$ and $b_{4i-2}$ is primitive, it occurs that

$$(\mathrm{Id}_{\Omega^4\boldsymbol{BO}}\wedge\kappa')^*\phi_{4,4}{}^*(w_{4i}) = \sum_{1\le k,1\le l,k+l=i} \binom{k+l-1}{l} Q_l\otimes\Sigma^2 b_{4k-2}$$

and

$$\phi_{4,6}{}^*(a_{4i-2}) = \sum_{1\le k,1\le l,k+l=j} \binom{k+l-1}{l} Q_l\otimes b_{4k-2}.$$

Remark that $\binom{k+l-1}{l} = \binom{4k+4l-4}{4l} = \binom{4k+4l-2}{4l}$ and

$$\phi_{4,6}{}^*(a_{2^p(4i-2)}) = \sum_{1\le k,1\le l,k+l=j} \binom{4k+4l-2}{4l} Q_l{}^{2^p}\otimes b_{4k-2}{}^{2^p}.$$

Therefore the statement is also true for $\phi_{4,6}$.

Q.E.D.(Theorem 4.2)

From Theorem 3.3 and Theorem 4.2, the next theorem follows.

**Theorem 4.7.** *Assume neither* $n-1$ *nor* $m-1$ *is a power of* 2 *and both* $n$ *and* $m$ *are odd. If* $\binom{n+m-2}{n-1}\equiv 0 \ mod\ 2$*,* $(n,m)$ *is Spin-regular.*

# 5 the case $n$ and $m$ are even

In this section we use integral cohomology. Consider the next diagram.

$$\begin{array}{ccccc}
S^{n-1} & \xrightarrow{\tilde{i}_n} & Spin(n) & \xrightarrow{\pi_n} & S^{n-1} \\
\downarrow p'_n & & \downarrow p_n & & \downarrow \cong \\
\mathrm{RP}^{n-1} & \xrightarrow{i_n} & SO(n) & \xrightarrow{\pi'_n} & S^{n-1}
\end{array}$$

Here $\pi_n$, $\pi'_n$ is the map obtained from $Spin(n) \to Spin(n)/Spin(n-1) = S^{n-1}$ and $SO(n) \to SO(n)/SO(n-1) = S^{n-1}$ respectively. Also $i_n$ is the inclusion map defined as follows. Let $l \in \mathrm{RP}^{n-1}$ be a line and let $e \in l$ be a unit vector. Then $i_n(l) = i'_n(l_0)i'_n(l)$ where $i'_n(l)(v) = v - 2(v,e)e$ and $l_0$ is the base point of $\mathrm{RP}^{n-1}$. We set $p'_n : S^{n-1} \to \mathrm{RP}^{n-1}$ be the usual covering map then there is a map $\tilde{i}_n$ which makes diagram commutative. Moreover, when $n = 4$, $\pi_n$ has a section $\epsilon : S^{n-1} \to Spin(n)$, that is, $\pi_n \circ \epsilon = \mathrm{Id}$.

We set $c_{n-1}$ as the generator of $\mathrm{H}^*(S^{n-1};\mathbb{Z})$ and take $\delta \in \mathrm{H}^*(Spin(n) \wedge Spin(m);\mathbb{Z})$ as $\delta = (\pi_n \wedge \pi_m)^*(c_{n-1} \otimes c_{m-1})$.

**Lemma 5.1.** *If $n$ and $m$ are even and neither $n$ nor $m$ is 4,*

$$\mathrm{H}^{n+m-2}(Spin(n) \wedge Spin(m);\mathbb{Z}) = \langle \delta \rangle \oplus Ker(\tilde{i}_n \wedge \tilde{i}_m)^*.$$

*Proof.* Since $n$ is even, ${i_n}^*{\pi'_n}^*(c_{n-1})$ is the generator of $\mathrm{H}^{n-1}(\mathrm{RP}^{n-1};\mathbb{Z}) \cong \mathbb{Z}$. Therefore

$$\tilde{i}_n^*{\pi_n}^*(c_{n-1}) = {p'_n}^*{i_n}^*{\pi'_n}^*(c_{n-1}) = 2c_{n-1}, \tag{12}$$

that is, ${\tilde{i}_n \wedge \tilde{i}_m}^*(\delta) = 4c_{n-1} \otimes c_{m-1}$.

Because ${p'_n}^* : \mathrm{H}^{n-1}(\mathrm{RP}^{n-1};\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{n-1}(S^{n-1};\mathbb{Z}/2\mathbb{Z})$ is a 0-map and $\tilde{i}_n^* \circ {p_n}^* = {p'_n}^* \circ {i_n}^*$, we have $\tilde{i}_n^* \circ {p_n}^* = 0$ in mod 2 cohomology. Further, since, when $n \neq 4$, ${p_n}^* : \mathrm{H}^{n-1}(SO(n);\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{n-1}(Spin(n);\mathbb{Z}/2\mathbb{Z})$ is epic, this implies that $\tilde{i}_n^* : \mathrm{H}^{n-1}(Spin(n);\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{n-1}(S^{n-1};\mathbb{Z}/2\mathbb{Z})$ is also a 0-map. Therefore $\mathrm{Im}\tilde{i}_n^* \subset \langle 2c_{n-1} \rangle$ in integral cohomology.

Now we obtain that $\mathrm{Im}(\tilde{i}_n \wedge \tilde{i}_m)^* = \langle 4c_{n-1} \otimes c_{m-1} \rangle = \langle (\tilde{i}_n \wedge \tilde{i}_m)^*(\delta) \rangle$ and from the freeness of $\mathrm{H}^{n+m-2}(S^{n+m-2};\mathbb{Z})$ the statement follows.

Q.E.D.

**Lemma 5.2.** *If $n = 4$ and $m$ are even and $m \neq 4$,*

$$\mathrm{H}^{n+m-2}(Spin(n) \wedge Spin(m);\mathbb{Z}) = \langle \delta \rangle \oplus Ker(\epsilon \wedge \tilde{i}_m)^*.$$

*Proof.* From (12) and $\epsilon^* \pi_4{}^*(c_3) = c_3$,

$$(\epsilon \wedge \tilde{i}_m)^*(\delta) = 2c_{n-1} \otimes c_{m-1}.$$

As seen in the proof of previous lemma, $\mathrm{Im}\tilde{i}_m^* \subset \langle 2c_{m-1} \rangle$ in integral cohomology and since $\epsilon$ is a section, $\mathrm{Im}\epsilon^* = \langle c_3 \rangle$.

Now it follows that $\mathrm{Im}(\epsilon \wedge \tilde{i}_m)^* = \langle 2c_3 \otimes c_{m-1} \rangle = \langle (\epsilon \wedge \tilde{i}_m)^*(\delta) \rangle$ and from the freeness of $\mathrm{H}^{n+m-2}(S^{n+m-2}; \mathbb{Z})$ the statement follows.

Q.E.D.

**Theorem 5.3.** *Assume neither $n-1$ nor $m-1$ is a power of 2, both $n$ and $m$ are even, $n+m \equiv 0$ mod 4 and $n+m \geq 16$. Then $(n,m)$ is Spin-regular.*

*Proof.* We use Proposition 2.2. Let $x : Spin(n) \wedge Spin(m) \to \Omega \boldsymbol{Spin}$ satisfies $x^*(\alpha_{n+m-2}) = x_{n-1} \otimes x_{m-1}$ in mod 2 cohomology. Then there exists $\eta \in \widetilde{KO}(\Sigma^2 Spin(n) \wedge Spin(m))$ which satisfies

$$w_{n+m}(\eta) = \Sigma^2 x_{n-1} \otimes x_{m-1}. \tag{13}$$

Here, since Pontrjagin square acts trivially in $\mathrm{H}^*(\Sigma^2 Spin(n) \wedge Spin(m); \mathbb{Z})$, by the second formula of Wu [12],

$$\rho_4(P_{\frac{n+m}{4}}(\eta)) = w'_{n+m}(\eta), \tag{14}$$

where $w'_{n+m}$ is the image of $w_{n+m}$ under the coefficient monomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$ and $\rho_4$ is the map of mod 4 reduction.

When neither $n$ nor $m$ is 4, from (13), (14) and Lemma 5.1, we can see that

$$P_{\frac{n+m}{4}}(\eta) = \Sigma^2((4k+2)\delta + \alpha),$$

where $\alpha \in \mathrm{Ker}(\tilde{i}_n \wedge \tilde{i}_m)^*$ and we obtain

$$P_{\frac{n+m}{4}}(\Sigma^2(\tilde{i}_n \wedge \tilde{i}_m)^*(\eta)) = (16k+8)c_{n+m}.$$

When $n = 4$ and $m \neq 4$, (13), (14) and Lemma 5.2 imply that

$$P_{\frac{n+m}{4}}(\eta) = \Sigma^2((4k+2)\delta + \beta),$$

where $\beta \in \mathrm{Ker}(\epsilon \wedge \tilde{i}_m)^*$ and we have

$$P_{\frac{n+m}{4}}(\Sigma^2(\epsilon \wedge \tilde{i}_m)^*(\eta)) = (8k+4)c_{n+m}.$$

But for the generator $\eta_0$ of $\widetilde{KO}(S^{n+m})$, $P_{\frac{n+m}{4}}(\eta_0)$ is divisible by $(\frac{n+m}{2} - 1)!$. [1] When $n + m \geq 16$ this is a contradiction and the statement follows.

Q.E.D.

**Theorem 5.4.** *Assume neither $n-1$ nor $m-1$ is a power of 2, both $n$ and $m$ are even. If $n+m=12$ or $n+m \equiv 2 \mod 4$ . Then $(n,m)$ is Spin-regular.*

*Proof.* We use Proposition 2.2. Let $x : Spin(n) \wedge Spin(m) \to \Omega\boldsymbol{Spin}$ be the arbitrary continuous map.

When $n+m \equiv 2 \bmod 4$, that is, $n+m-2$ is divisible by 4, $x^*(\alpha_{n+m-2}) = x^*(\alpha_{\frac{n+m-2}{2}})^2$ in mod 2 cohomology. Thus $x^*(\alpha_{n+m-2})$ can be written in the form $\sum \alpha \otimes \beta$ where $\alpha$ and $\beta$ are decomposable. Therefore $x^*(\alpha_{n+m-2}) \neq x_{n-1} \otimes x_{m-1}$.

Now let $n+m=12$ and $n \leq m$. When $n \neq 4$, $x^*(\alpha_6) = x_3 \otimes x_3$ or 0 and when $n=4$, $x^*(\alpha_6) = z \otimes x_3$, $x_3 \otimes x_3$ or 0. We can see

$$\mathrm{Sq}^2 x^*(\alpha_6) = x^*(\mathrm{Sq}^2 \alpha_6) = x^*(\alpha_8) = x^*(\alpha_2)^4 = 0$$

while

$$\mathrm{Sq}^2 x_3 \otimes x_3 = x_5 \otimes x_3 + x_3 \otimes x_5,$$

$$\mathrm{Sq}^2 z \otimes x_3 = z \otimes x_5.$$

So $x^*(\alpha_6) = 0$ and we have

$$x^*(\alpha_{10}) = x^*(\mathrm{Sq}^4 \alpha_6) = \mathrm{Sq}^4 x^*(\alpha_6) = 0.$$

Q.E.D.

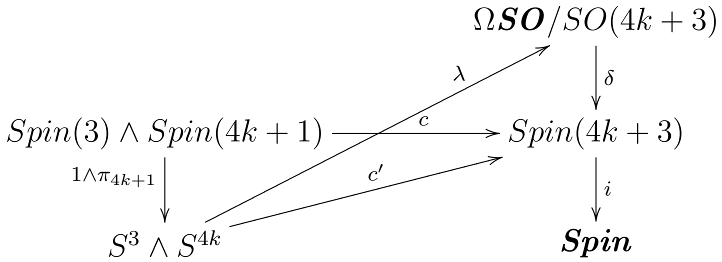
From Proposition 2.2, Theorems 4.7, 5.3, 5.4, we finally obtain Theorem 1.3.

# 6 $(3, 4k+1)$ is Spin-irregular

In this section we shall give the proof of Theorem 1.4 which requires that $(3, 4k+1)$ is Spin-irregular.

Since there are embeddings $Spin(3) \to Spin(4k+3)$, $Spin(4k+1) \to Spin(4k+3)$ where any element of $Spin(3)$ and any element of $Spin(4k) \subset Spin(4k+1)$ exactly commute in $Spin(4k+3)$. Let $A \in Spin(3)$, $B \in Spin(4k+1)$, $C \in Spin(4k) \subset Spin(4k+1)$. Then $A(BC)A^{-1}(BC)^{-1} = ABCA^{-1}C^{-1}B^{-1} = ABA^{-1}B^{-1}$ and the commutator of $A$ and $B$ is invariant under the right translation of $Spin(4k)$ on $B$.

Therefore there exists a map $c' : Spin(3) \wedge (Spin(4k+1)/Spin(4k)) \to Spin(4k+3)$ such that $c' \circ (1 \wedge \pi_{4k+1}) \simeq c$. See the diagram below. Remark that $Spin(3) \cong S^3$ and $Spin(4k+1)/Spin(4k) \cong S^{4k}$.




In the above diagram $\Omega\boldsymbol{SO}/SO(4k+3) \to Spin(4k+3) \to \boldsymbol{Spin}$ is a fibration and $i \circ c'$ is null homotopic. So there exists a map $\lambda : S^{4k+3} \to \Omega\boldsymbol{SO}/SO(4k+3)$, such that $\delta \circ \lambda \simeq c'$.

Since $\pi_{4k+4}(\boldsymbol{SO}/SO(4k+3)) \cong 0$ ([10]), $\pi_{4k+3}(\Omega\boldsymbol{SO}/SO(4k+3)) \cong 0$ and $\lambda$ is null homotopic.

Thus $c \simeq \delta \circ \lambda \circ (1 \wedge \pi_{4n+1}) \simeq *$ and Theorem 1.4 is proved.

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